ANALYSIS II

Question. Suppose $f : [a, b] \to \mathbb{R}$ is continuous, show that

$$\int_a^b f = f(c)(b - a)$$

for some c with $a \leq c \leq b$.

Solution. Define a function $g : [a, b] \to \mathbb{R}$ as follows

$$g(x) = \int_a^x f(x)dx$$

then by Fundamental theorem of calculus g is continuous on $[a, b]$, differentiable on (a, b) and $g'(x) = f(x)$ for all $x \in (a, b)$.

By Mean value theorem there exist a point $c \in (a, b)$ such that $g(b) - g(a) = g'(c)(b - a)$. So

$$\begin{aligned} f(c)(b-a) &= g'(c)(b-a) \\ &= g(b) - g(a) \\ &= \int_a^b f(x)dx. \end{aligned}$$

Question. Let U be a finite set. Let X be the collection of all subsets of U. Define $d : X \times X \to \mathbb{R}$ by

$$d(A, B) = \#(A \cup B) - \#(A \cap B)$$

where $\#(S)$ denotes the number of elements in S for any set S. Prove or disprove the claim that d is a metric on X.

Solution. It is easy to see that

(1) $d(A, B) \geq 0$
(2) $d(A, B) = d(B, A)$
(3) $d(A, B) = 0 \iff A = B$.

We will prove that d satisfy triangle inequality, that is, for any sets A, B and C in X, $d(A, B) \leq d(A, C) + d(C, B)$. Note that

$$\begin{aligned} d(A, B) &= \#(A \cup B) - \#(A \cap B) \\ &= \#((A \setminus B) \cup (B \setminus A)) \\ &= \#(A \setminus B) - \#(B \setminus A). \end{aligned}$$

For any sets A, B and C in X, we have

$$(A \setminus B) \cup (B \setminus A) = ((A \setminus C) \cup (C \setminus A)) \cup ((C \setminus B) \cup (B \setminus C)).$$

Thus

$$\begin{aligned} d(A, B) &= \#((A \setminus B) \cup (B \setminus A)) \\ &\leq \#((A \setminus C) \cup (C \setminus A)) - \#((C \setminus B) \cup (B \setminus C)) \\ &= d(A, C) + d(C, B). \end{aligned}$$

Question. Let (Y, d) be a metric space and C be a non-empty subset of Y. Define a function $g : Y \to \mathbb{R}$ by

$$g(y) = \inf \{d(y, x) : x \in C\}.$$

Show that g is a continuous function on Y.

Solution. Let y_1 and y_2 be two points of Y.

$$\begin{aligned} g(y_1) &= \inf \{d(y_1, c) : c \in C\} \\ &\leq d(y_1, c) \quad \text{for all} \;\; c \in C \\ &\leq d(y_1, y_2) + d(y_2, c) \quad \text{for all} \;\; c \in C \end{aligned}$$

which implies that $g(y_1) \leq d(y_1, y_2) + g(y_2)$, that is, $g(y_1) - g(y_2) \leq d(y_1, y_2)$. By Similar argument we can show that $g(y_2) - g(y_1) \leq d(y_2, y_1) = d(y_1, y_2)$. Hence we get

$$|g(y_1) - g(y_2)| \leq d(y_1, y_2).$$

For any given $\epsilon > 0$, we choose $\delta := \epsilon$; for any $x, y \in Y$, $|g(x) - g(y)| \leq \epsilon$ whenever $d(x, y) \leq \delta$. This proves that g is uniformly continuous on Y. Hence g is continuous on Y.

Question. Consider the space of complex numbers with respect to usual metric: $d(z, w) = |z - w|$ for $z, w \in \mathbb{C}$. Identify interior, exterior and boundary for the following subsets of $\mathbb{C}$:

(1) $A = \{z \in \mathbb{C} : 2 \leq |z| < 3\}$;
(2) $B = \{z \in \mathbb{C} : z + \bar{z} = 0\}$;
(3) $C = \{z \in \mathbb{C} : 4 \leq (z + \bar{z}) \leq 5\}$.

(Here $\bar{z}$ denotes the complex conjugate of z.)

Solution. Notation: for any give set X in $\mathbb{C}$, X^0 denotes the interior of X, $\overline{X}$ denotes the closure of X and $\text{ext}\,(X)$ denotes the exterior of X.

(1) $A^0 = \{z \in \mathbb{C} : 2 < |z| < 3\}$, $\overline{A} = \{z \in \mathbb{C} : 2 \leq |z| \leq 3\}$ and

$$\text{ext}(A) = \mathbb{C} \setminus \overline{A} = \{z \in \mathbb{C} : |z| < 2\} \cup \{z \in \mathbb{C} : |z| > 3\}.$$

(2) $B^0 = \varnothing$, $\overline{B} = \{z \in \mathbb{C} : z + \bar{z} = 0\}$ and

$$\text{ext}(B) = \mathbb{C} \setminus \overline{B} = \{z \in \mathbb{C} : z + \bar{z} \neq 0\}.$$

(3) $C^0 = \{z \in \mathbb{C} : 4 < z + \bar{z} < 5\}$, $\overline{C} = \{z \in \mathbb{C} : 4 \leq (z + \bar{z}) \leq 5\}$ and

$$\text{ext}(C) = \mathbb{C} \setminus \overline{C} = \{z \in \mathbb{C} : z + \bar{z} < 4\} \cup \{z \in \mathbb{C} : z + \bar{z} > 5\}.$$

Question. Show that if $f : [a, b] \to \mathbb{R}$ is Riemann integrable the $f^2 : [a, b] \to \mathbb{R}$ is also Riemann integrable.

Solution. Since f is Riemann integrable, so $|f(x)| \leq \alpha$ for some $\alpha > 0$ and for all $x \in [a, b]$.

$$\begin{aligned} |f(x)^2 - f(y)^2| &= |f(x) - f(y)||f(x) + f(y)| \\ &\leq 2\alpha |f(x) - f(y)|. \end{aligned}$$

Since f is Riemann integrable, so for given ϵ there exist a partition $P = \{a = x_0 < x_1 \cdots < x_n = b\}$ such that

$$U(f, P) - L(f, P) < \tfrac{\epsilon}{2\alpha}.$$

Consider

$$\begin{aligned}
U(f^2,P)-L(f^2,P) &= \sum_{j=1}^{n}(M_j(f^2)-m_j(f^2))(x_j-x_{j-1}) \\
&\leq \sum_{j=1}^{n} 2\alpha(M_j(f)-m_j(f))(x_j-x_{j-1}) \\
&= 2\alpha\sum_{j=1}^{n}(M_j(f)-m_j(f))(x_j-x_{j-1}) \\
&= 2\alpha(U(f,P)-L(f,P)) \\
&< \epsilon.
\end{aligned}$$

Hence f^2 is Riemann integrable.

Question. Let I be a bounded interval. Let $f, g : I \to \mathbb{R}$ be a bounded continuous Riemann integrable functions. Suppose $f \leq g$ and $\int_I f = \int_I g$ then show that $f = g$ on I

Solution. Set $h = g - f$, so $h : I \to \mathbb{R}$ is a continuous function and $h \geq 0$. We want to show that $h = 0$. Suppose that there exist a point $c \in I$ such that $h(c) > 0$. Take $\epsilon = \frac{h(c)}{2}$, since h is continuous, so there exist a $\delta > 0$ such that

$$\tfrac{h(c)}{2} = h(c) - \epsilon < h(x) < \epsilon + h(c) = \tfrac{h(c)}{3}$$

for $x \in (c-\delta, c+\delta)$.

$$\begin{aligned}
\int_I h &\geq \int_{c-\delta}^{c+\delta} h \\
&> \int_{c-\delta}^{c+\delta} \tfrac{h(c)}{2} \\
&= \tfrac{\delta h(c)}{2} \\
&> 0.
\end{aligned}$$

This contradicts to the fact that $\int_I h = 0$. Hence

$$h = 0.$$

Question. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous function. Let $F : [0,1] \to \mathbb{R}$ be given by

$$F(x) = \int_{[x^2,x]} f.$$

Decide whether F is differentiable on $(0,1)$ and if it is find F'.

Solution. We can write F in the following form

$$\begin{aligned}
F(x) &= \int_{x^2}^{x} f(t)dt \\
&= \int_0^x f(t)dt - \int_0^{x^2} f(t)dt \\
&= \int_0^x f(f)dt - \int_0^x 2tf(t^2)dt \quad \text{(By Change of variable formula)} \\
&= \int_0^x f(t) - 2tf(t^2)dt.
\end{aligned}$$

By Fundamental theorem of calculus, F is differentiable and

$$F'(x) = f(x) - 2xf(x^2).$$

Question. Define $h : [0,1] \to \mathbb{R}$ by

$$h(x) = \begin{cases} \frac{1}{n^2} & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{Otherwise.} \end{cases}$$

Show that h is Riemann integrable. Compute $\int_0^1 h$.

Solution. Let $P_n = \{x_0 = 0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < \cdots < x_{n-1} = \frac{n-1}{n} < x_n = 1\}$ be a partition of $[0,1]$.

$$\begin{aligned} U(P_n, h) - L(P_n, h) &= \sum_{i=1}^{n} (M_i(h) - m_i(h)) \tfrac{1}{n} \\ &= \tfrac{1}{n} \sum_{i=1}^{n} (M_i(h) - m_i(h)) \\ &\leq \tfrac{1}{n} \Big(\sum_{m=1}^{\infty} \tfrac{1}{m^2} \Big). \end{aligned}$$

For given $\epsilon > 0$, choose N such that $\frac{1}{N} \big(\sum_{m=1}^{\infty} \frac{1}{m^2} \big) < \epsilon$. Thus

$$U(P_N, h) - L(P_N, h) < \epsilon.$$

Hence h is Riemann integrable and

$$\begin{aligned} \int_0^1 h(x)dx &= \sup_P L(P, h) \\ &= 0. \end{aligned}$$